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ADAPTIVE HIERARCHIC MODELLING OF PLATES AND SHELLS WITH A-POSTERIORI ERROR ESTIMATION

by

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The problem of a-posteriori estimation of the modelling error and the adaptive modelling of plates is addressed in this paper. A family of models is proposed and upper and lower a-posteriori estimates of the modelling error in various norms are provided. The estimates are locally asymptotically exact.

ADAPTIVE HIERARCHICAL MODELLING OF PLATES AND SHELLS WITH A-POSTERIORI ERROR ESTIMATION

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Summary. The paper addresses the problem of a-posteriori estimation of the modelling error and the adaptive modelling of plates. It proposes a family of models and provides upper and lower a-posteriori estimates of the modelling error in various norms . The estimates are locally asymptotically exact.

1. Introduction

The reliability of computational engineering analysis depends on the *reliability of the underlying mathematical formulation* (i.e. the mathematical model) of the engineering problem under consideration as well as on the *reliability of the numerical treatment* of the mathematical model.

The *selection of the mathematical formulation is the most crucial aspect of reliability in the analysis of an engineering problem.*

Reality can never be completely identified with any mathematical formulation. Nevertheless, in practice reality is identified with the exact solution of a complex mathematical problem to which we shall refer as "*true problem*" throughout. Prior to its computational solution, however, the true problem is approximated by a structurally simpler, so-called *mathematical model* and then the computational analysis is performed on this selected mathematical model. This means that the reliability of the computed solution should not be measured against the mathematical model, but rather against the *true problem*, i.e. it should take into account the discretization *and* the modelling error. (For a discussion of mathematical modelling in engineering we refer to [1]).

The usual engineering approach is to test the reliability of a mathematical model by a set of benchmark analyses for which the solution of the true problem is known. This leads often to the design of *various models* tailored to different objectives of the engineering analysis.

No single mathematical model can be good (or optimal) under all circumstances of the engineering analysis and therefore a decision has to be made, upon which particular model the computational analysis should be based. The strategy of the model selection must be governed by an *a-posteriori* assessment of the modelling error and an adaptive selection of the model (without knowledge of the exact solution of the true problem). The derivation of reliable a-posteriori estimators for the modelling error is the topic of the present paper. More specifically, we will discuss the hierarchical modeling of thin structures, such as plates and shells, in mechanics.

2. The plate and shell problems and their hierarchical models.

The problem of plate-and shell analysis has been in the forefront of engineering interest for a long time. Here the *true* problem is the *three-dimensional problem of linear elasticity* on a thin domain - a planar or curved surface of thickness d. Plate-and shell models are then certain 2-dimensional problems, i.e. the modelling leads to a reduction of the dimension of the problem by one.

A particular model still widely used today was proposed already in the first half of the nineteenth century by S. Germain [2] and G. Kirchhoff [3]. The derivation of this and other models is mostly based on physical considerations or mathematical analyses of various degrees of rigor (see e.g. [4]-[19] and the references therein) or on asymptotic analysis of the three-dimensional, true problem as the thickness d tends to 0 (see, for example [9], [20]-[28] and the references therein). Other results address the theoretical relation between the true problem and the models (see e.g. [29] - [32]) or they investigate the derivation of hierarchical families of models as general problem of dimensional reduction (see e.g. [33] - [36] for related ideas).

The solutions of various proposed models could lead to significantly different results which explain the very large number of proposed formulations in the literature today. For theoretical and numerical analyses of various models see [37], [38]. Let us consider as an example for the difference between various models the clamped square plate $\Omega^d :=$ $(-0.5,0.5)^2 \times (-0.005,0.005)$ of thickness 0.01 with uniform load. We will assume the Young's modulus of elasticity $E = 10^7$ and a Poisson ratio $\nu = 0.3$. Let us be interested in the bending moments $M_{11}(x_1,0)$ and $M_{22}(x_1,0)$. Table 1 shows the data for the three dimensional ("true") formulation, the Reissner-Mindlin model (RM) and the model (1,1,2) (see [37], [39], [40] for the definition of (1,1,2) model).

We see that the reliability of the model strongly depends on the aim of the analysis. If we are interested in $M_{11}(0.4,0)$, $M_{22}(0.4,0)$ then the difference between the models is practically negligible but in the case $M_{22}(0.5,0)$ the difference is significant (30%)

Table 3.1. The moments $M_{11}(x_1,0)$ and $M_{22}(x_1,0)$ for various models.

x_1	$M_{11}(x_1,0)$			$M_{22}(x_1,0)$		
	3 dim	RM	(1,1,2)	3 dim	RM	(1,1,2)
0.	-0.0229	-0.0229	-0.0229	-0.0229	-0.0229	-0.0229
0.2000	-0.0157	-0.0157	-0.0157	-0.0163	-0.0163	-0.0163
0.4000	-0.0164	-0.0163	-0.0164	-0.0026	-0.0027	0.0026
0.4900	0.0470	0.0164	0.0470	0.0141	0.0141	0.0141
0.4990	0.0509	0.0509	0.0509	0.0168	0.0152	0.0170
0.4993	0.0510	0.0510	0.0510	0.0176	0.0153	0.0179
0.4999	0.0512	0.0512	0.0512	0.0207	0.0153	0.0212
0.5000	0.0513	0.0513	0.0513	0.0220	0.0153	0.0220

3. <u>The boundary value problem and the hierarchy of the models</u>.

To present all the essential ideas of adaptive modelling in the simplest setting, we will restrict ourselves to the problem of the Laplace equation on a thin, plate-like domain. Nevertheless the approach presented here can be used in general [41].

Let $\omega \subset \mathbb{R}^2$ denote a bounded domain with piecewise smooth boundary Γ and define the *plate of thickness* d, $\Omega^d := \{x = (x_1,x_2,x_3) \in \mathbb{R}^3 \mid (x_1,x_2) \in \omega, |x_3| < d/2\}$. Further, we introduce the *lateral boundary* S and the *faces of the plate* $R_\pm$:

$$S := \{x \in \mathbb{R}^3 \mid (x_1,x_2) \in \Gamma, |x_3| < d/2\}, \quad R_\pm := \{x \in \mathbb{R}^3 \mid (x_1,x_2) \in \omega, x_3 = \pm d/2\}.$$

We will be interested in the problem

$$\begin{aligned} \Delta w &= 0 \quad \text{on } \Omega^d, \\ w &= 0 \quad \text{on } S, \\ \frac{\partial w}{\partial n} &= \frac{1}{2}f \quad \text{on } R_\pm. \end{aligned} \tag{3.1}$$

Here $f(x_1,x_2) \in L^2(\omega)$ and n denotes the exterior unit normal vector.

Let us cast (3.1) into the weak form. Denote by

$$H := \{u \in H^1(\Omega^d) \mid u = 0 \text{ on } S\} \tag{3.2}$$

and define by

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$$B(u,v) := \int_{\Omega^d} \nabla u \cdot \nabla v \; dv \tag{3.3}$$

the bilinear form on $H \times H$. Further, let

$$F(v) := \frac{1}{2} \int_{\omega} f(x_1,x_2)(v(x_1,x_2,d/2) + v(x_1,x_2,-d/2))dx_1 dx_2. \tag{3.4}$$

Then the weak form of (3.1) reads:

Find $u_0 \in H$ such that

$$B(u_0,v) = F(v) \quad \forall v \in H. \tag{3.5}$$

Obviously here the solution is symmetric, i.e.

$$u(x_1,x_2,x_3) = u(x_1,x_2,-x_3). \tag{3.6}$$

Assuming that the thickness d of Ω^d is small, we will approximate the full, three dimensional problem (3.1) - (3.6) by a sequence of two dimensional problems, the *hierarchical family of plate models*. Let us define these models.

Denote by $\mathcal{P} = \{\omega_i | \omega_i \subseteq \omega, \; 1 \le i \le n\}$ a partition of ω into n domains ω_i with piecewise smooth boundaries $\partial\omega_i$ such that $\omega_i \cap \omega_j = \phi$ for $i \ne j$ and $\bar{\omega} = \bigcup_{i=1}^{n} \bar{\omega}_i$.

We will allow the order q_i of the models to be different in different subdomains and define

$$q := \{q_1,q_2,\cdots,q_n\}, \; q_i \ge 0 \text{ integers.}$$

Denoting by L_j the Legendre polynomial of degree j on $(-1,1)$, we introduce the space

$$S(\mathcal{P},q) := \left\{ u \in H | u|_{\omega_i} = \sum_{j=0}^{q_i} U_j^{(i)}(x_1,x_2)L_j\left[\frac{2x_3}{d}\right], \omega_i \in \mathcal{P} \right\}$$

and define the $(\mathcal{P},q)$-*plate model* as the boundary value problem:

Find $u(\mathcal{P},q) \in S(\mathcal{P},q)$ such that

$$B(u(\mathcal{P},q),v) = F(v) \quad \forall v \in S(\mathcal{P},q). \tag{3.7}$$

Obviously, due to (3.6), we can assume a priori that $U_j^{(i)} = 0$ for odd j. The solution $u(\mathcal{P},q)$ of the $(\mathcal{P},q)$-plate model is the (energy) projection of the three dimensional solution u_0 of (3.5) onto $S(\mathcal{P},q)$.

We have

Theorem 3.1. Let $e = u_0 - u(\mathcal{P},q)$. Then

$$\int_{-d/2}^{d/2} e(x_1,x_2,x_3)dx_3 = 0 \tag{3.8}$$

almost everywhere.

Proof For any $U(x_1,x_2) \in S(\mathcal{P},q)$ we have $B(e,U) = 0$. Observing that $\frac{\partial u_0}{\partial n} = 0$ on $R_\pm$ and integrating by parts we get (3.8).

Now we state the goal of the computation (and of the hierarchical modelling):

Given load data f, *a norm* $|\circ|$ *on* H, *and a tolerance* $\tau > 0$, *find a model* $(\mathcal{P},q)$ *so that*

$$|u_0 - u(\mathcal{P},q)| < \tau. \tag{3.9}$$

Remark 3.1. If any reasonable norm $|\circ|$, such that $|u_0| < \infty$, and the fixed thickness d of the plate are given, then such a model exists: Select in (3.7) $n = 1$ and $\mathcal{P} = \{\omega\}$, $q = \{q\}$ sufficiently large. □

Remark 3.2. We assumed here that the $(\mathcal{P},q)$ plate model can be solved exactly. This is, of course, not true in practice, since typically (3.7) must also be solved numerically by e.g. the finite element method. We distinguish between the modelling error (3.7) and the discretization error. Since we are here primarily interested in the modelling error, we assume below that the discretization error is negligible, i.e. that the $(\mathcal{P},q)$ model is solved to sufficiently high accuracy. □

Remark 3.3. All the notions introduced above apply also verbatim to the plate problem discussed in Section 2. The model problem (3.1) - (3.7) exhibits the basic characteristics of the plate- and shell problems. □

Remark 3.4. The selection $S(\mathcal{P},q)$ in (3.7) is not arbitrary. It was shown in [33], [34] that this selection has, for fixed $\mathcal{P}$, two essential properties, namely:

a) the selection of polynomials in x_3 is the only choice which leads to the optimal rate of convergence as $d \longrightarrow 0$ (although for fixed $d > 0$ this choice is not optimal [34]),

b) as $q_i \longrightarrow \infty$, $1 \le i \le n$, $u(\mathcal{P},q) \longrightarrow u_0$ and a-priori error estimates are given in [33], [34].

For laminated beams and plates, an alternative subspace $S(\mathcal{P},q)$ with similar properties is known [17], [19]. □

Remark 3.5.

The adaptive selection of the model has now two essential parts:

i) Design of the local error indicator, which leads for the norm under considerations, to the error estimator,

ii) Design of the adaptive procedures.

We will address these points in the next sections. □

4) Some abstract results

Let H_1, H_2 be two reflexive Banach spaces furnished with the norms $\|\cdot\|_1$ and $\|\cdot\|_2$, respectively. Further, let $B(u,v)$ be a bilinear form defined on $H_1 \times H_2$. We will call the bilinear form (C,γ)-*regular* if there exist constants $0 < C$, $\gamma < \infty$ so that

(4.1) $$|B(u,v)| \le C\|u\|_1\|v\|_2,$$

(4.2) $$\inf_{\|u\|_1=1} \sup_{\|v\|_2=1} |B(u,v)| \ge \gamma,$$

(4.3) $$\text{for any } v \ne 0,\ v \in H_2, \quad \sup_{\|u\|_1=1} |B(u,v)| > 0.$$

Bilinear forms satisfying (4.1) - (4.3) have the following properties (see [42]).

a) Let $f \in (H_2)'$ (i.e. f is a bounded, linear functional on H_2), then there exists exactly one $u \in H_1$ such that

$$B(u,v) = f(v), \quad \forall v \in H_2.$$

b) If

$$\sup_{\|v\|_2=1} |B(u,v)| \le A, \tag{4.4}$$

then

$$\|u\|_1 \le \frac{A}{\gamma}.$$

Let us consider now some special cases which will be important later.

<u>Theorem 4.1</u>. Let

$$H_1 = H_2 = \left\{ u \in H \mid \int_{-d/2}^{d/2} u(x_1, x_2, x_3)dx_3 = 0 \quad \text{almost everywhere} \right\} \tag{4.5}$$

where H is as in (3.2). Let further

$$\|u\|_1 = \left[\iint_{\Omega^d} |\nabla u|^2 dx \right]^{1/2} = \|u\|_2. \tag{4.6}$$

Then the bilinear form

$$B(u,v) := \int_\Omega \nabla u \cdot \nabla v \; dx \tag{3.3}$$

is (C,γ)-regular with $C = \gamma = 1$. □

<u>Theorem 4.2</u>. Let $\varphi(x_1,x_2) > 0$, $(x_1,x_2) \in \omega$ have bounded first derivatives and define

$$Q := \max_{\substack{(x_1,x_2)\in\omega \\ i=1,2}} \left|\frac{\partial\varphi^2}{\partial x_i}\right| \Big/ \varphi^2(x_1,x_2). \tag{4.7}$$

Further let $H_1^\varphi = H_1$, $H_2^\varphi = H_2$ where H_1 and H_2 are the same spaces as before, but, instead of $\|\circ\|_1$ and $\|\circ\|_2$, furnished with the norms $\|\circ\|_{1,\varphi}$ and $\|\circ\|_{2,\varphi}$ respectively, which are given by

$$\|u\|_{1,\varphi}^2 = \int_{\Omega^d} \varphi^2 |\nabla u|^2 dx, \tag{4.8}$$

$$\|v\|_{2,\varphi}^2 = \int_{\Omega^d} \varphi^{-2} |\nabla v|^2 dx = \|v\|_{1,\varphi^{-1}}^2. \tag{4.9}$$

Then the bilinear form (3.2) is (C,γ)-regular with $C = 1$ and,

$$\gamma \ge \gamma_0 = \left(1 - 2\frac{d}{\pi}Q\right)^{1/2} \left(1 + \frac{d}{\pi}Q\left(2 + Q\frac{d}{\pi}\right)\right)^{-1/2}. \tag{4.10}$$

Proof. By the Schwarz inequality we see immediately that (4.1) holds with $C = 1$.

Now, let $u \in H_1^{\varphi}$, $\|u\|_{1,\varphi}=1$ and define $v = \varphi^2 u$. Then $v \in H_2^{\varphi}$ and

$$\frac{\partial v}{\partial x_i} = \varphi^2 \frac{\partial u}{\partial x_i} + u\frac{\partial \varphi^2}{\partial x_i}, \quad i = 1,2,$$

$$\frac{\partial v}{\partial x_3} = \varphi^2 \frac{\partial u}{\partial x_3} .$$

Hence

$$B(u,v) = \|u\|^2_{1,\varphi} + \int_{\Omega} \left[\sum_{i=1}^{2} \frac{\partial u}{\partial x_i} u \frac{\partial \varphi^2}{\partial x_i} \right] dx. \tag{4.11}$$

Realizing that $\int_{-d/2}^{d/2} u(x_1,x_2,x_3)dx_3 = 0$ almost everywhere, we have

$$\int_{-d/2}^{d/2} u^2(x_1,x_2,x_3)dx_3 \le \left(\frac{d}{\pi}\right)^2 \int_{-d/2}^{d/2} \left[\frac{\partial u}{\partial x_3}(x_1,x_2,x_3)\right]^2 dx_3. \tag{4.12}$$

Hence, with (4.7) and $i = 1,2$ we have for any $\varepsilon > 0$

$$\left| \int_{\Omega^d} \frac{\partial u}{\partial x_i} u \frac{\partial \varphi^2}{\partial x_i} dx \right| \le \frac{1}{2}Q\left[\varepsilon \int_{\Omega^d} \left(\frac{\partial u}{\partial x_i}\right)^2 \varphi^2(x_1,x_2)dx\right.$$

$$\left. + \frac{1}{\varepsilon}\frac{d^2}{\pi^2} \int_{\Omega^d} \left(\frac{\partial u}{\partial x_3}\right)^2 \varphi^2(x_1,x_2)dx\right] \le \frac{1}{2}Q\left(\varepsilon + \frac{d^2}{\pi^2}\varepsilon^{-1}\right) \|u\|^2_{1,\varphi}.$$

Selecting $\varepsilon = \frac{d}{\pi}$, we get

$$\left| \int_{\Omega^d} \left[\sum_{i=1}^{2} \frac{\partial u}{\partial x_i} u \frac{\partial \varphi^2}{\partial x_i} \right] dx \right| \le 2\frac{d}{\pi}Q\|u\|^2_{1,\varphi}$$

and

$$|B(u,v)| \geq \|u\|^2_{1,\varphi}\left[1 - 2\frac{d}{\pi}Q\right].$$

Similarly

$$\|v\|^2_{2,\varphi} = \|u\|^2_{1,\varphi} + \int_{\Omega^d}\left[2\sum_{i=1}^{2}\frac{\partial u}{\partial x_i}u\frac{\partial\varphi^2}{\partial x_i} + u^2\left(\frac{\partial\varphi^2}{\partial x_i}\right)\right]^2\varphi^{-2}dx \leq \|u\|^2_{1,\varphi}\left[1 + \frac{d}{\pi}Q\left[2 + Q\frac{d}{\pi}\right]\right].$$

Hence

$$1 \geq \gamma \geq \left[1 - 2\frac{d}{\pi}Q\right]^{1/2}\left[1 + \frac{d}{\pi}Q\left[2 + Q\frac{d}{\pi}\right]\right]^{-1/2}.$$

Since $B(u,v)$ is symmetric, (4.3) holds and the theorem is proven. □

We can select in particular any $\varphi(x_1, x_2)$ so that $2Q \leq \frac{\pi}{d} \cdot \alpha$, $0 < \alpha < 1$, and get

$$\gamma \geq (1-\alpha)^{1/2}(1 + \alpha(1+\alpha))^{-1/2}.$$ □

Since $\frac{\partial\varphi^2}{\partial x_i} = 2\varphi\frac{\partial\varphi}{\partial x_i}$ we have

$$Q := 2 \max_{\substack{(x_1,x_2)\in\omega \\ i=1,2}} \left|\frac{\partial\varphi}{\partial x_i}\right| \Big/ |\varphi|. \tag{4.13}$$

Of particular interest in sections 5 and 6 will be the weight function

$$\varphi = e^{-\beta(|x_1 - x_1^0| + |x_2 - x_2^0|)} \tag{4.14}$$

where $Q = 2\beta$. Here we can select $4\beta = \frac{\pi}{d}\alpha$, $0 < \alpha < 1$ and obtain that $B(\cdot,\cdot)$ is (C,γ)-regular on $H_1^\varphi \times H_2^\varphi$.

We note that for $0 < \kappa_1 < \varphi < \kappa_2$ we get by a simple argument

$$\gamma \geq \frac{\kappa_1}{\kappa_2}.$$

Let us now consider the bilinear form

$$B_1(u,v) = \int_{\Omega^d} u\Delta v \, dx \tag{4.15}$$

defined on $\overset{*}{H}_1 \times \overset{*}{H}_2$ where

$$H_1^* = \left\{ u \in L_2(\Omega) \mid \int_{-d/2}^{d/2} u(x_1,x_2,x_3)dx_3 = 0 \quad \text{almost everywhere} \right\},$$

furnished with the norm

$$\|u\|_1 = \left[\int_{\Omega^d} |u|^2 dx \right]^{1/2} = \|u\|_{L_2(\Omega^d)}$$

and where

$$H_2^* = \left\{ v \in H_1(\Omega^d) \mid \|\Delta v\|_{L_2(\Omega^d)} < \infty, \ \frac{\partial v}{\partial n} = 0 \quad \text{on} \quad R_\pm \right\}$$

where H is defined in (4.5), furnished with the norm

$$\|v\|_2 = \left[\int_{\Omega^d} |\Delta v|^2 dx \right]^{1/2} = \|\Delta v\|_{L_2(\Omega^d)}.$$

We note that locally $v \in H^2(\Omega^d)$ and hence $\frac{\partial v}{\partial n}$ makes sense. It also follows easily that $\|\cdot\|_2$ is a norm on H_2^*.

It is easy to see that (4.1) holds with $C = 1$. We will now estimate γ.

For given $u \in H_1^*$ let us define s to be solution of the equation

(4.16) $$\Delta s = u,$$

(4.17) $$s = 0 \quad \text{on} \quad S,$$

(4.18) $$\frac{\partial s}{\partial n} = 0 \quad \text{on} \quad R_\pm.$$

Because $u \in L_2(\Omega)$, s obviously exists and is uniquely determined. Define

(4.19) $$z = s - \frac{1}{d} \int_{-d/2}^{d/2} s(x_1,x_2,x_3)dx_3$$

then also (4.17) and (4.18) holds for z.

Further, by using (4.18)

$$\Delta z = \Delta s - \frac{1}{d} \int_{-d/2}^{d/2} (\Delta s)(x_1,x_2,x_3)dx_3,$$

hence

$$\Delta z = u - \frac{1}{d}\int_{-d/2}^{d/2} u(x_1,x_2,x_3)dx_3 = u.$$

Therefore

$$B_1(u,z) = \int_{\Omega^d} u\Delta z dx = |||u|||_1^2, \quad |||\Delta z|||_2 = |||u|||_1$$

and (4.2) follows with $\gamma = 1$.

Hence we have proved

Theorem 4.2. The bilinear form $B_1(u,v)$ defined by (4.14) on $H_1^* \times H_2^*$ satisfies (4.1) and (4.2) with $C = \gamma = 1$. □

5. A posteriori estimates of the modelling error.

In this section we assume that the $(\mathcal{P},q)$-plate model is given and its exact solution $u(\mathcal{P},q)$ defined in (3.7) is known. We will be interested in estimating the modelling error

$$|u_0 - u(\mathcal{P},q)|$$

for a particular norm $|\circ|$ which is L^p-based. With the norm $|\circ|$ we associate the *error indicator function* $\eta(x_1,x_2)$ so that, for $1 < p < \infty$,

$$(5.1) \qquad \mathcal{E}(u(\mathcal{P},q)) = \left[\int_\omega |\eta(x_1,x_2)|^p dx_1 dx_2\right]^{1/p} \approx |u_0 - u(\mathcal{P},q)|.$$

The quantity $\mathcal{E}$ will be called *error-estimator*. The *effectivity index* Θ corresponding to $\mathcal{E}$ is defined by

$$(5.2) \qquad \Theta := \frac{\mathcal{E}(u(\mathcal{P},q))}{|u_0 - u(\mathcal{P},q)|}.$$

We say that $\mathcal{E}$ is an *upper (lower) estimator,* if $\Theta > 1 (\Theta < 1)$, respectively, and call the estimator $\mathcal{E}$ (κ_1,κ_2)*-proper with respect to a class* T *of data* f, *if*

$$(5.3) \qquad 0 < \kappa_1 \le \Theta \le \kappa_2 < \infty$$

holds for all data $f \in T$.

The estimator $\mathcal{E}$ is *asymptotically* respectively *spectrally exact* on T, if

$$(5.4) \qquad \Theta \longrightarrow 1 \ \text{ as } \ d \longrightarrow 0^+ \ \textit{ respectively as } \ q \longrightarrow \infty,$$

for u_0 belonging uniformly in d to certain classes of data.

We shall say that the indicator function $\eta(x_1,x_2)$ is *locally asymptotically exact* (resp. *locally proper*), if (5.4) (resp. (5.3)) holds for the norm $|\circ| = \|\circ\|_{1,\varphi}$ where the weight is given by

$$(5.5) \qquad \varphi(x_1,x_2) := \exp \frac{\alpha}{d^\rho}\left[|x_1 - x_1^0| + |x_2 - x_2^0|\right], \ \rho \in (0,1), \ 0 < \alpha$$

and $\left[x_1^0, x_2^0\right] \in \omega$ arbitrary.

We will address here for simplicity only the case $n = 1$, $p = 2$, $\mathcal{P} = \{\omega\}$ and $q = \{q\}$, $q \geq 0$, and denote the corresponding subspace by $S(q)$.

Throughout this section we will denote by $\|\circ\|_{1,\varphi}$ the norm defined in (4.8) and the norms of f are always understood as integrals over ω only. Let us first analyze the modelling error in this norm. We begin by analyzing the best possible asymptotic rate of convergence of the $(\mathcal{P},q)$-model.

<u>Theorem 5.1</u> Assume that $q \geq 0$ is an integer and

$$(5.6) \qquad \Delta^{-1+j} f \in (\mathring{H}^1 \cap H^2)(\omega) \qquad \text{for } \ 1 \leq j \leq q.$$

Then there exists $\bar{u}(2q) \in S(2q)$, $u_0 - \bar{u}(2q) \in H_1$ (see (4.5)) such that

$$(5.7) \qquad \|u_0 - \bar{u}(2q)\|_{1,\varphi} \leq C_q d^{2q+1/2} \|\Delta^q f\|_{0,\varphi}$$

<u>Proof</u>: We only sketch it since it follows closely that of [33, Theorem 3.1]. First observe that for $\gamma \geq \gamma_0$ as in (4.10)

$$(5.8) \qquad \gamma_0 \|u_0 - \bar{u}(2q)\|_{1,\varphi} \leq \sup_{\|v\|_{2,\varphi} \neq 0} \frac{|B(u_0 - u(2q), v)|}{\|v\|_{2,\varphi}} .$$

Now, in [33] it was shown that there exists a sequence of polynomials ψ_{2j} of degree $2j$, $j = 0,1,2,\ldots$ which is independent of d such that

$$\bar{u}(2q) := \sum_{j=0}^{q} d^{-1+2j}\left(\Delta^{j-1}f\right)\psi_{2j}\left(\frac{2x_3}{d}\right)$$

satisfies, for every $v \in H$,

$$B(u_0 - \bar{u}(2q), v) = d^{2q+1} \int_{-d/2}^{d/2} \int_{\omega} \frac{\partial r_{2q}}{\partial x_3} \frac{\partial v}{\partial x_3}\, dx_1 dx_2 dx_3$$

where $r_{2q}(x_1, x_2) = \psi_{2q+2}\left(\frac{2x_3}{d}\right)(\Delta^q f)(x_1, x_2)$. Now (5.7) is a simple consequence of Schwartz's inequality and of (5.8). □

Remark 5.1 If $q = 0$ then $f \in L^2(\omega)$ is sufficient for (5.7) to hold. □

Remark 5.2 By definition (3.7) of the $(\mathcal{P}, q)$ model and Theorem 3.1 the approximation is *quasioptimal*, i.e.

$$(5.9) \qquad \|u_0 - u(\mathcal{P}, q)\|_{1,\varphi} \le C(\gamma_0) \inf_{\chi \in S(\mathcal{P}, q)} \|u_0 - \chi\|_{1,\varphi}. \qquad □$$

To derive the error indicator function $\eta(x_1, x_2)$, we note that using Theorem 3.1 the *modelling error* $e_{2q} := u_0 - u(\mathcal{P}, q)$ belongs to $H_\perp$ defined in (4.5) and satisfies

$$(5.10) \qquad B(e_{2q}, v) = R_{2q}(v), \quad \forall v \in H$$

where the space H was defined in (3.2) and where

$$(5.11) \qquad \begin{aligned} R_{2q}(v) = {} & \frac{1}{2} \int_{\omega} r(x_1, x_2)(v(x_1, x_2, d/2) + v(x_1, x_2, -d/2))dx_1 dx_2 \\ & + \int_{\omega} \int_{-d/2}^{d/2} v(x_1, x_2, x_3)\Delta u(\mathcal{P}, q)dx_3 dx_1 dx_2. \end{aligned}$$

Since $B(e_{2q}, v) = 0$ for all $v \in S(2q)$, we can restrict v in (5.10) to be of the form

$$(5.12) \qquad \int_{-d/2}^{d/2} v(x_1, x_2, x_3)dx_3 = 0,$$

for a.e. $(x_1,x_2)\in\omega$. Consequently, we arrive at

$$(5.11)' \qquad 0 = R_{2q}(v) = \frac{1}{2}\int_\omega r(x_1,x_2)(v(x_1,x_2,d/2) + v(x_1,x_2,-d/2))dx_1dx_2$$

$$+\int_\omega\int_{-d/2}^{d/2} v(x_1,x_2,x_3)\Delta u(\mathcal{P},q)dx_1dx_2dx_3,$$

for all $v\in S(2q)$ where

$$(5.13a) \qquad r(x_1,x_2) = f(x_1,x_2) - 2\frac{\partial u(\mathcal{P},q)}{\partial n}(x_1,x_2,d/2).$$

$$(5.13b) \qquad \Delta u(\mathcal{P},q) = \sum_{j=0}^{q} A_{2j}(x_1,x_2)L_{2q}\left[\frac{2x_3}{d}\right].$$

<u>Remark 5.3</u> The condition (5.12) shows, together with Theorems 3.1 and 4.2, that the bilinear form in (5.10) can also be understood as

$$B:\ H_1^\varphi\times H_2^\varphi \longmapsto \mathbb{R}.$$

This implies in particular that $e_{2q}\in H_1^\varphi$ for all $q\ge 0$.

Along the same lines, B can also be interpreted as a (C,γ) regular form on $S_1^\varphi\times S_2^\varphi$ where

$$S_i^\varphi(q) := S(q)\cap H_i^\varphi,\quad i = 1,2,$$

and it is clear that

$$u(2q_1) - u(2q_2)\in S_1^\varphi(q_2)$$

if $0\le q_1<q_2$. This is essential for the adaptive procedure. □

From (5.8) we have that

$$\gamma_0\|e_{2q}\|_{1,\varphi} \le \sup_v \frac{R_{2q}(v)}{\|v\|_{2,\varphi}} \le \sup_v \frac{R_{2q}(v)}{\left[\int_\omega\int_{-d/2}^{d/2}\varphi^{-2}\left(\frac{\partial v}{\partial x_3}\right)^2(x_1,x_2,x_3)dx_1dx_2dx_3\right]^{1/2}}$$

and with (5.11)' and Theorem 4.2 we find that

$$\gamma_0\|e_{2q}\|_{1,\varphi} \le \int_\omega \varphi^2 \sup_v \left\{ \frac{\frac{1}{2}r(x_1,x_2)\left[v(x_1,x_2,d/2)+v(x_1,x_2,-d/2)\right]}{\int_{-d/2}^{d/2}\left(\frac{\partial v}{\partial x_3}\right)^2 dx_3} \right.$$

(5.14)

$$\left. + \sum_{j=0}^{q} A_{2j}(x_1,x_2)\, \frac{\int_{-d/2}^{d/2} L_{2j}\left(\frac{2x_3}{d}\right)v(x_1,x_2,x_3)dx_3}{\int_{-d/2}^{d/2}\left(\frac{\partial v}{\partial x_3}\right)^2 dx_3} \right\} dx_1dx_2$$

and the supremum is taken over all $0 \ne v \in H$ which satisfy (5.12). It is easy to see that the supremum in (5.14) is attained on a function v which satisfies (5.12) and

$$-\frac{\partial^2 v}{\partial x_3^2}(x_1,x_2,x_3) = \sum_{j=0}^{q} A_{2j}(x_1,x_2)L_{2j}\left(\frac{2x_3}{d}\right),$$

$$\frac{\partial v}{\partial x_3}(x_1,x_2,\pm d/2) = \mp \frac{1}{2}r(x_1,x_2),$$

where $A_{2j}(x_1,x_2)$ is completely determined by (5.11)'. Hence

(5.15) $$v(x_1,x_2,x_3) = \frac{1}{2}L_{2q+1}\left(\frac{2x_3}{d}\right)r(x_1,x_2).$$

Since

(5.16) $$\int_{-d/2}^{d/2}\left[L_{2q+1}\left(\frac{2x_3}{d}\right)\right]^2 dx_3 = \frac{d}{2}\frac{1}{(4q+3)}$$

we find that in (5.14)

(5.17) $$\gamma_0^2\|e_{2q}\|_{1,\varphi}^2 \le dC_q \int_\omega r^2(x_1,x_2)\varphi^2(x_1,x_2)dx_1dx_2,$$

where

(5.18) $$C_q = \frac{1}{4(4q+3)} .$$

Based on the estimate (5.17) we define the error indicator function by

(5.19) $$\eta_{2q}^2(x_1,x_2) := dC_q r^2(x_1,x_2)\varphi^2(x_1,x_2).$$

Remark 5.4. We emphasize that η_{2q} is very easy to compute, especially for low order models. In particular, we find from (5.13) that for $q = 0$

(5.20) $$\eta_0^2(x_1,x_2) = \frac{d}{12} f^2(x_1,x_2)\varphi^2(x_1,x_2),$$

i.e. the indicator does not depend on $u(0)$ and can be computed a-priori. □

From (5.14) and (5.19) we readily obtain

$$\|e_{2q}\|_{1,\varphi}^2 \le \frac{1}{\gamma_0^2} \int_\omega \eta_{2q}^2 dx_1 dx_2$$

and hence

(5.21) $$\|e_{2q}\|_{1,\varphi} \le \frac{1}{\gamma_0} \mathcal{E}(u(\mathcal{P},q)).$$

Hence $\mathcal{E}$ based on (5.19) is an upper estimator and its effectivity index Θ is bounded from below by γ_0.

Let us show that $\mathcal{E}$ also bounds the error from below. This will allow us then to prove that $\mathcal{E}$ is locally asymptotically exact for any q. Since the modelling error e_{2q} satisfies

$$B(e_{2q},v) = \frac{1}{2} \int_\omega r(x_1,x_2)\Big(v(x_1,x_2,d/2) + v(x_1,x_2,-d/2))$$

$$+ \int_{-d/2}^{d/2} \sum_{j=0}^{q} A_{2j}(x_1,x_2)L_{2j}\left(\frac{2x_3}{d}\right)v(x_1,x_2,x_3)dx_3\Big)dx_1dx_2$$

for all $v \in H$ so that (5.12) holds, we select $v = \bar{v}\varphi^2$ with $\bar{v}$ as in (5.15) and obtain

(5.22) $$dC_q \int_\omega r^2\varphi^2 dx_1, dx_2 = B(e_{2q},v) \le \|e_{2q}\|_{1,\varphi}\|v\|_{2,\varphi}.$$

An elementary calculation shows with $\nabla_x v = \left(\frac{\partial v}{\partial x_1}, \frac{\partial v}{\partial x_2}\right)$ that

$$|\nabla_x v|^2 \le \varphi^2\{3\varphi^2|\nabla_x \bar{v}|^2 + 6|\bar{v}|^2|\nabla_x \varphi|^2\},$$

hence

$$(5.23)\qquad \|v\|^2_{2,\varphi} = \int_{-d/2}^{d/2}\int_\omega \left\{\varphi^{-2}|\nabla_x v|^2 + \varphi^{-2}\left[\frac{\partial v}{\partial x_3}\right]^2\right\}dx_1dx_2dx_3.$$

$$\le \int_{-d/2}^{d/2}\int_\omega \left\{3\varphi^2|\nabla_x\bar{v}|^2 + 6|\bar{v}|^2|\nabla_x\varphi|^2 + \varphi^{-2}\left|\frac{\partial v}{\partial x_3}\right|^2\right\}dx_1dx_2dx_3$$

$$(5.23)\qquad \le 3d^3C_q^2D_q\left[|\!|\!|r|\!|\!|^2_{1,\varphi} + Q^2|\!|\!|r|\!|\!|^2_{0,\varphi}\right] + dC_q|\!|\!|r|\!|\!|^2_{0,\varphi}$$

where C_q is defined by (5.16), Q by (4.13) and

$$(5.24)\qquad D_q = 2\,\frac{4q+3}{(4q+3)^2-4}$$

and we used the notation

$$(5.25)\qquad |\!|\!|r|\!|\!|^2_{1,\varphi} = \int_\omega |\nabla_x r|^2\varphi^2 dx_1dx_2,\quad |\!|\!|r|\!|\!|^2_{0,\varphi} = \int_\omega r^2\varphi^2dx_1dx_2.$$

Now we estimate in (5.22) for $\varepsilon > 0$

$$dC_q|\!|\!|r|\!|\!|^2_{0,\varphi} \le \frac{\varepsilon}{2}\,\|e_{2,q}\|^2_{1,\varphi} + \frac{1}{2\varepsilon}\,\|v\|^2_{2,\varphi}.$$

If we select $\varepsilon_0 > 0$ so that

$$(5.26)\qquad \frac{1}{2\varepsilon_0}\|v\|^2_{2,\varphi} \le \frac{1}{2}dC_q|\!|\!|r|\!|\!|^2_{0,\varphi} = \frac{1}{2}\mathcal{E}^2(u(\mathcal{P},q))$$

we get the desired bound

$$dC_q|\!|\!|r|\!|\!|^2_{0,\varphi} = \mathcal{E}^2(u(\mathcal{P},q)) \le \varepsilon_0\|e_{2q}\|^2_{1,\varphi}.$$

Using (5.23) and (5.26) we find

$$(5.27)\qquad \varepsilon_0 = \left\{1 + 3d^2C_qD_q\left[Q^2 + \frac{|\!|\!|r|\!|\!|^2_{1,\varphi}}{|\!|\!|r|\!|\!|^2_{0,\varphi}}\right]\right\}$$

and we have

<u>Theorem 5.2</u>. Assume that $f \in T = T_\beta := \{f|$ either $r(q) = 0$ or $|\!|\!|r|\!|\!|_{1,\varphi}/|\!|\!|r|\!|\!|_{0,\varphi} \le \beta < \infty\}$. Then the estimator $\mathcal{E}$ based on (5.19) is locally (κ_1,κ_2)-proper with

$$\kappa_1 = \gamma_0 = \left[1 + \frac{d}{\pi}Q(2 + Q\frac{d}{\pi})\right]^{-1/2}\left[1 - 2\frac{d}{\pi}Q\right]^{1/2}$$

(see (4.10)) and with

$$\kappa_2 = \{1 + 3C_q D_q d^2(\beta^2 + Q^2)\}^{1/2}. \tag{5.28}$$

Here C_q and D_q are explicitly given in (5.18) and (5.24). If $\varphi = 1$, then $Q = 0$ and the factor 3 in (5.28) can be dropped.

Remark 5.5: With the weight φ as in (5.5) and $\alpha = 1$, we find easily that $Q \le d^{-\rho}$, $0 < \rho < 1$. Hence, for $f \in T_{\bar{\beta}d^{-\rho}}$ for some $\bar{\beta}$ independent of d, the estimator $\mathcal{E}$ is locally *asymptotically* (as $d \longrightarrow 0$) exact. □

Remark 5.6: From (5.18) and (5.24) we see that $\kappa_2 \longrightarrow 1$ as $q \longrightarrow \infty$ for $f \in T_\beta$, i.e., $\mathcal{E}$ based on (5.19) is also locally *spectrally* exact. □

Remark 5.7. The local asymptotic exactness of $\mathcal{E}$ not only insures that the indicator function η_{2q} in (5.17) gives a good overall estimate $\mathcal{E}$, but also that the local indicator

$$\mathcal{E}_{2q}(\rho, x_1^0, x_2^0) = \left[dC_q \int_{|x_1 - x_1^0| + |x_2 - x_2^0| < d^\rho} r^2\ (x_1, x_2)dx_1dx_2\right]^{1/2} \tag{5.29}$$

is an asymptotically exact measure for the *local contribution* to the modelling error. This is an essential ingredient for the adaptive selection of the model order. □

Remark 5.8 Our analysis is obviously also valid for $\omega = (-1,1)$ and we present now an example to demonstrate the sharpness of the bounds in Theorem 5.2. For $\Omega^d = (-1,1) \times (-d/2, d/2)$, let us select f in (3.1) so that

$$u(x,y) = 2\cos\left[\frac{\pi}{2}x\right]\cosh\left[\frac{\pi}{2}y\right].$$

Then

$$u(q) = \sum_{j=0}^{q} U_j(x)L_{2j}\left[\frac{2y}{d}\right] = \sum_{j=0}^{q} x_j\cos\left[\frac{\pi}{2}x\right]L_{2j}\left[\frac{2y}{d}\right].$$

The vector $\underline{x} = (x_0, \cdots, x_q)^T$ is determined from

$$\left[\frac{d\pi}{8}\underline{A} + \frac{2}{d}\underline{B}\right]\underline{x} = \alpha\underline{e}$$

where

$$\alpha = 2\pi \sinh\left(\frac{\pi d}{4}\right), \quad e = (1,\cdots,1)^T$$

and

$$\underline{A}_{ij} = \int_{-1}^{1} L_{2i}L_{2j}dz, \quad \underline{B}_{ij} = \int_{-1}^{1} L'_{2i}L'_{2j}dz.$$

We find explicitly for the weight $\varphi = 1$

$$\|e_{2q}\|^2_{1,\varphi} = \alpha\left(2\cosh\left(\frac{\pi d}{4}\right) - \underline{x}^T\underline{e}\right)$$

and the estimator

$$\mathcal{E}^2 = dC_q\left(\alpha - \frac{4}{d}\sum_{i=1}^{q} x_j L'_{2j}(1)\right)^2.$$

Using a computer algebra system, we obtained

$$(\Theta(q))^2 = \mathcal{E}^2/\|e_{2q}\|^2_{1,\varphi} = 1 + d^2\frac{\pi^2}{m_q} + O(d^4)$$

where m_q is listed in Table 5.1.

q	0	1	2	3	4	5	6
m_q	240	360	936	1768	2856	4200	5800

Table 5.1 m_q in the asymptotic expansion of the effectivity index

Not only is the estimator $\mathcal{E}$ asymptotically exact as predicted in Theorem 5.2, but we observe that with $Q = 0$ and $\beta = \frac{\pi}{2}$ the ratio of $(\kappa_2)^2 - 1$ in (5.28) and π^2/m_q in the expansion of $\Theta(q)$ is 1, i.e.

$$\kappa_2 = \left\{1 + C_q D_q \beta^2 d^2\right\}^{1/2}$$

is the best possible value for $\varphi = 1$.

Let us now analyze the error estimator for the norm $\|e\|_{L_2(\Omega^d)} = \|e\|_1$. To this end we consider the bilinear form (4.15) and get

(5.30) $$B_1(e_{2q},v) = R_{2q}(v)$$

with $R_{2q}(v)$ given in (5.11)'.

Hence, using Theorem 4.2, we get

$$\|e\|_{L_2(\Omega^d)} = \sup \frac{|R_{2q}(v)|}{\|\Delta v\|_{L_2(\Omega^d)}}$$

where the supremum is taken over all $0 \neq v \in H_2^*$ which satisfy (5.12).

To estimate the supremum, we observe that any $v \in H_2^*$ can be written in the form

(5.31)
$$v(x_1,x_2,x_3) = \left(\frac{2}{d}\right)^{1/2} \sum_{k,\ell\geq 1} a_{k\ell}\varphi_k(x_1,x_2)\cos\left(\frac{2\ell\pi x_3}{d}\right) + \left(\frac{2}{d}\right)^{1/2} \sum_{k\geq 1} a_{k0}\varphi_k(x_1,x_2)$$

where $\varphi_k(x_1,x_2)$ denote the eigenfunctions (orthonormalized in $L^2(\omega)$) of the eigenvalue problem

$$-\Delta\varphi_k = \lambda_k\varphi_k, \quad \varphi_k = 0 \text{ on } \partial\omega$$

and $\{\lambda_k\}_{k>1}$ is the corresponding sequence of positive eigenvalues. Note that $R_{2q}(v)$ is as in (5.11)' and, as before, $A_{2j}(x_1,x_2) = -(4j+1)\ r\ (x_1,x_2)/d$. Further, since $R_{2q}(v) = 0$ for any $v(x_1,x_2,x_3) = \varphi(x_1,x_2)$, $\varphi(x_1,x_2) = 0$ on $\partial\omega$, we can omit the second term in the expression (5.31) for v and find

$$-\Delta v = \left(\frac{2}{d}\right)^{1/2} \sum_{k,\ell\geq 1} b_{k\ell}\varphi_k(x_1,x_2)\cos\left(\frac{2\ell\pi x_3}{d}\right)$$

where

$$b_{k\ell} = a_{k\ell}\left(\lambda_k + \left(\frac{2\ell\pi}{d}\right)^2\right).$$

Inserting (5.31) into (5.11)', we find

$$R_{2q}(v) = \left(\frac{2}{d}\right)^{1/2} \int_\omega \left\{ r(x_1,x_2) \sum_{k,\ell\geq 1} a_{k\ell}[1-\Lambda_{q\ell}]\varphi_k(x_1,x_2)\right\} dx_1 dx_2$$

where

$$\Lambda_{q\ell} = \frac{1}{2} \sum_{j=1}^{q} (4j+1) \int_{-1}^{1} L_{2q}(z)\cos(\ell\pi z)dz$$

(and $\Lambda_{0\ell} = 0\ \forall\ell$), and we estimate

$$(R_{2q}(v))^2 \le \|r\|^2_{L^2(\omega)} \frac{2}{d} \sum_{k\ge1} c_k^2$$

where

$$c_k = \sum_{\ell\ge1} a_{k\ell}[1-\Lambda_{q\ell}] = \sum_{\ell\ge1} \frac{b_{k\ell}[1-\Lambda_{q\ell}]}{\lambda_k + \left[\frac{2\ell\pi}{d}\right]^2},$$

which implies with Schwartz' inequality that

$$\sum_{k\ge1} c_k^2 \le \left[\frac{d}{2\pi}\right]^4 \left[\sum_{\ell\ge1} \frac{(1-\Lambda_{q\ell})^2}{\ell^4}\right] \left[\sum_{k,\,\ell\ge1} b_{k\ell}^2\right].$$

Altogether hence

$$\|e_{2q}\|^2_{L^2(\Omega^d)} = \sup_{H_2} \frac{(R_{2q}(v))^2}{\|\Delta v\|^2_{L^2(\Omega^d)}} \le d^3 E_q \|r\|^2_{L^2(\omega)} \tag{5.32}$$

where

$$E_q = \frac{1}{8\pi^4} \sum_{\ell\ge1} \frac{(1-\Lambda_{q\ell})^2}{\ell^4}.$$

<u>Remark 5.9</u>. For $q = 0$, we have $\Lambda_{q\ell} = 0$ and hence $E_0 = \zeta(4)/(8\pi^4) = 1/720$. For $q > 0$, E_q can be easily computed numerically. The bound (5.32) is better than the estimate

$$\|e_{2q}\|^2_{L^2(\Omega^d)} \le d^3 \frac{C_q}{\pi^2} \|r\|^2_{L^2(\omega)}$$

which is obtained from (5.21) with $\varphi = 1$ and from

$$\|e_{2q}\|_{L^2(\Omega^d)} \le \frac{d}{\pi} |e_{2q}|_{H^1(\Omega^d)}.$$

The estimate (5.32) is in fact optimal, as is demonstrated in Table 5.2.

q	$d^3\|r\|^2_{L^2(\omega)} \;/\; \|e_{2q}\|^2_{L^2(\Omega^d)}$
0	$720 + (\pi d)^2/60480 \quad + \mathcal{O}(d^4)$
1	$2520 + (\pi d)^2/221760 \quad + \mathcal{O}(d^4)$
2	$10296 + 1287(\pi d)^2/35 \quad + \mathcal{O}(d^4)$
3	$26520 + 9945(\pi d)^2/209 + \mathcal{O}(d^4)$

Table 5.2. Asymptotics of L^2-residual versus L^2-error for small q for the example in Remark 5.8.

In all cases the numerical value of the leading term was found to be equal to that of $1/E_q$ which suggests the asymptotic and spectral exactness of the L^2 a-posteriori estimator

$$\mathcal{E}_{L2} = d^{3/2}\sqrt{E_q}\,\|r\|_{L^2(\omega)}. \qquad \square$$

6. Adaptive Selection of the local model order.

We consider now the $(\mathcal{P},q)$-model based on an arbitrary partition $\mathcal{P} = \{\omega_i\}_{1\le i\le n}$ as described in Section 3, where $q = \{q_1,\cdots,q_n\}$ is the vector of model orders on ω_i. In the previous section we showed that the local contribution to the modeling error corresponding to ω_i can be reliably estimated by a calculation involving the local residuals only. This gives rise to the following simple *feedback procedure* to adjust the model orders.

(6.1)

i) given a parameter $0 < \lambda < 1$ and an order vector q, compute the local indicators η_i on the domains ω_i according to (5.19).

ii) For the $n\lambda$ of the largest error indicators η_i^2 raise q_i by one and solve again.

iii) Stop when $\mathcal{E} \le \tau$.

The strategy (6.1) is analogous to one version of adaptive finite element methods. We emphasize, however, that (6.1) is not optimal since often q_i has to be increased by more than one and the computational solution of the current $(\mathcal{P},q)$-model is costly.

Let us present another adaptive strategy. We start with the observation that for the error $e(\mathcal{P},q)$ we have

$$B(e,v) = R(v) \ \forall v \in H. \tag{6.2}$$

By Remark 5.3, we know that $e \in H_1^{\varphi}$, and, selecting φ suitably, it follows that e depends only on the local residual on $R_{\pm}$. We can, therefore, obtain an (asymptotically exact) approximation to e on $\Omega_i^d := \omega_i \times (-d/2, d/2)$ by solving (6.2) approximately on spaces of functions which vanish on $\Omega^d \setminus \bar{\Omega}_i^d$. We emphasize that these problems are completely decoupled and can be solved very cheaply and in parallel. Therefore, we can assume that the indicators $\eta_i^2(q_i)$ are known for several values of q_i and all $\omega_i \in \mathcal{P}$.

There are many ways to optimize models from the hierarchy of $(\mathcal{P},q)$ models. Any concept of optimality involves cost, e.g. we assume that the work in the numerical solution of the $(\mathcal{P},q)$-model is given by

$$W(\mathcal{P},q) = \sum_{i=1}^{n} F_i(q_i) \tag{6.3}$$

where the F_i are monotonically increasing functions depending, e.g., on the computer structure, implementation, etc. Then we can, based on a given model $u(\mathcal{P},q)$, find an optimal order vector $\tilde{q}$ by $\eta_i^2(\tilde{q}_i)$ by solving

$$\min \sum_{i=1}^{n} F_i(q) \quad \text{subject to} \quad \sum_i \eta_i^2(\tilde{q}_i) = \tau^2. \tag{6.4}$$

where τ is given tolerance. This problem has at least one solution. Moreover, using a Langrange multiplier λ and assuming that $q \in \mathbb{R}$, for simplicity, we find that at the optimum necessarily

$$\sum_{i=1}^{n} \eta_i^2(\tilde{q}_i) = \tau^2 \tag{6.5}$$

$$\frac{d}{dq}\left[\eta_i^2(\tilde{q}_i)\right] = \lambda \frac{dF_i}{dq}(\tilde{q}_i), \tag{6.6}$$

or, using difference quotients, that q should satisfy

$$(6.7)\qquad \frac{\eta_i^2(q_i^{opt}) - \eta_i^2(q_i^{opt}-1)}{F_i(q_i^{opt}) - F_i(q_i^{opt}-1)} = \lambda, \quad 1 \le i \le n.$$

This is a weighted equilibration condition characterizing the optimal $(\mathcal{P},q)$-model. Once we have determined $\tilde{q}$ from (6.4) we solve for $u(\mathcal{P},\tilde{q})$ and possibly repeat the process with improved estimates for the η_i^2. The strategy is capable of predicting directly the optimal distribution q and uses very few iterations. Under suitable assumptions the orders q_i found in this way are very close to the best possible ones, hence the modelling is an adaptive one (i.e. a feedback procedure with certain optimality properties).

We presented here only the key ideas and refer to [41] for details and a more rigorous analysis.

7. Adaptive modeling in mechanics

We have shown in the previous sections the main ideas of a-posterori error estimators and the adaptive modeling. Although we have addressed only a most simple model problem, ideas of this type are applicable in general (for more see [39]).

Various models have different properties of the solution. For example, they have different boundary layer behavior, different singularities in the neighborhood of the corners, etc.

We emphasize that the estimators introduced above are *not* asymptotical in nature. They provide estimates for domains with a thickness which is not small. It is essential that these estimates are available for various norms, because in practice the aims of the modeliing differ.

Let us also mention that the ideas explained briefly above are applicable to many different model - formulations, for example, to models based on mixed methods etc. Various formulations also behave differently when they are descretized by the finite element method. As an example we mention to the locking problem [43].

We have assumed here that the "true problem" was the three-dimensional problem for completely specified data. In engineering, however, often the available data have intrinsic *uncertainties*. These effects have to be treated by a proper formulation of the true problem which accounts for these uncertainties, such as, for example, a stochastic one.

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